Cooperative behavior of qutrits with dipole-dipole interactions

A. Mandilara and V. M. Akulin

Laboratoire Aimé Cotton, Bât. 505, CNRS II, Campus d'Orsay, ORSAY CEDEX F-91405, FRANCE

We have identified a class of many body problems with analytic solution beyond the mean-field approximation. This is the case where each body can be considered as an element of an assembly of interacting particles that are translationally frozen multi-level quantum systems and that do not change significantly their initial quantum states during the evolution. In contrast, the entangled collective state of the assembly experiences an appreciable change. We apply this approach to interacting three-level systems.

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The description of the collective behavior of quantum ensembles beyond the mean-field approximation is one of the most challenging tasks of modern physics. The experiments with three-level Rydberg atoms performed during last decade [1, 2, 3] unambiguously point out the important role of essentially many-body phenomena in frozen gases with interaction among the internal degrees of freedom [4]. Besides, the three-level systems (qutrits) have been considered [5] in the context of quantum informatics where, in particular, the questions of generalized entanglement [6] and coherence protection [7] have been addressed recently. Therefore, analytical results considering dynamics of quantum assemblies [8] of three-level elements beyond the mean-field approach and hence allowing for the multipartite entanglement have become an issue of general interest.

Here we present the exhaustive description of an important particular class of quantum states of an assembly of N interacting identical qutrits where each single qutrit state remains close to the initial state with predominantly populated middle level. Still the collective assembly quantum state, being essentially entangled, differs from a product state typical of the mean-field approximation and moreover it can considerably deviate from the initial state. Significant deviation of the collective state which occurs in spite of small deviations of the single-particle states is known in the many-body theory as the orthogonality catastrophe. Our approach can be generalized to assemblies of arbitrary multilevel elements that remain close to their initial states.

For the description we employ a technique of nilpotentials inspired by the ideas of the Glauber coherent states [9], which has been adapted for three-level systems and employed for the description of quantum entanglement [10]. The collective state under consideration

$$|\Psi_W\rangle = \mathrm{e}^{\sum_{i,j=1}^{N} u_j^+ W_{ij} t_i^+} |\mathsf{O}\rangle \tag{1}$$

can be represented in terms of the commuting operators t_i^+ and u_i^+ defined as the nilpotent $su(3)$ operators [12] that create the upper $|1\rangle_i = t_i^+ |0\rangle_i$ and the lower $|-1\rangle_i = u_i^+ |0\rangle_i$ states, respectively, by acting on the middle state $|0\rangle_i$ of i-th qutrit. Here $|\mathsf{O}\rangle = \prod_i |0\rangle_i$ denotes the initial state of the assembly where all qutrits are in the middle state [11]. The requirement *(i)* that each qutrit is close to the initial state implies $|W| = \mathrm{Tr}WW^+ \ll N$. The assembly state (1) is not normalized to unity, although the entanglement matrix W_{ij} contains all information about the state, including normalization. It is expedient to explicitly give the normalization factor $\langle \Psi_W | \Psi_W\rangle$ and the population n_1 of the upper states of qutrits $|1\rangle_i$

$$\langle \Psi_W | \Psi_W\rangle = \exp\{\mathrm{Tr}WW^+\},\ n_1 = \mathrm{Tr}\frac{WW^+}{1 - WW^+} \tag{2}$$

corresponding to the state vector eq.(1). It is also worth mentioning that the sum $f = \Sigma_{i,,j=1}^{N} u_j^+ W_{ij} t_i^+$ represents the tanglemeter [10] of the state $|\Psi_W\rangle$. Due to our initial assumption f is not of the most generic form,i.e. it is lacking terms like $\Sigma_{i,,j=1}^{N} u_j^+ A_{ij} u_i^+$ or $\Sigma_{i,,j=1}^{N} t_j^+ B_{ij} t_i^+$. Nevertheless f contains cross terms which ensure, according to the entanglement criterion [10], the existence of entanglement among the qutrits of the assembly under consideration.

We now consider an assembly of qutrits with dipole-dipole interaction $\sum_{i,<j} V_{ji} \widehat{d}_j \widehat{d}_i$ and find the time-dependent matrix $\widehat{W}$. For the dipole moment operator $d_i = u_i^+ + t_i^- + u_i^- + t_i^+$ the interaction Hamiltonian reads

$$\widehat{H}_{int} = \sum_{i \neq, j} (u_j^+ + t_j^-) V_{ji} (u_i^- + t_i^+). \tag{3}$$

We assume that the single qutrit Hamiltonian has the form

$$\widehat{H}_0 = \beta\lambda^{(3)} + \alpha\frac{\lambda^{(8)}}{\sqrt{3}} = \begin{pmatrix} \beta + \frac{\alpha}{3} & 0 & 0 \\ 0 & -\frac{2\alpha}{3} & 0 \\ 0 & 0 & -\beta + \frac{\alpha}{3} \end{pmatrix}, \tag{4}$$

and that the coupling matrix V_{ji} has a spectral decomposition $V_{ji} = \sum_m C_{jm} V_m C_{mi}$. We also assume that $\beta \gg \alpha$ and exclude the high frequency β in the rotating wave approximation. Then the evolution operator in the interaction representation can be written as a functional integral

$$\exp\left\{-\mathrm{i}\widehat{H}t\right\} = \frac{1}{A}\int \mathrm{e}^{-\mathrm{i}S} \prod_m \mathcal{D}Z_m(t)\mathcal{D}Z_m^*(t), \tag{5}$$

where the normalization constant A and the action

$$S = \int \left[\sum_m Z_m(t) Z_m^*(t) - \sum_{m,j} V_m^{1/2} C_{jm} (\mathrm{e}^{\mathrm{i}\alpha t} u_j^+ \right.$$

$$+\mathrm{e}^{-\mathrm{i}\alpha t}t_j^-)Z_m(t)$$
$$\left.-\sum_{m,j}V_m^{1/2}C_{mj}(\mathrm{e}^{-\mathrm{i}\alpha t}u_j^-+\mathrm{e}^{\mathrm{i}\alpha t}t_j^+)Z_m^*(t)\right]\mathrm{d}t$$

are given in terms of the components $Z_m(t)$ and $Z_m^*(t)$ of two complex conjugated vector-function variables.

The representation (5) allows one to consider dynamics of qutrits independently: each qutrit now is subject to an action of the single-particle time-dependent Hamiltonian

$$\widehat{H}_i\,(t)=(\mathrm{e}^{\mathrm{i}\alpha t}u_i^+ +\mathrm{e}^{-\mathrm{i}\alpha t}t_i^-)E_i(t)+(\mathrm{e}^{-\mathrm{i}\alpha t}u_i^- +\mathrm{e}^{\mathrm{i}\alpha t}t_i^+)E_i^*(t) \tag{6}$$

where

$$E_i(t)=\sum_m V_m^{1/2}C_{im}Z_m(t) \tag{7}$$

is an "effective electric field" depending on the functional variables $Z_m(t)$. The requirement *(i)* justifies the employment of the second-order time-dependent perturbation theory expression

$$\widehat{U}_i(t)\simeq 1-\mathrm{i}\int^t\widehat{H}_i\,(x)\,\mathrm{d}x-\int^t\widehat{H}_i\,(y)\int^y\widehat{H}_i\,(x)\,\mathrm{d}x\mathrm{d}y \tag{8}$$

for the evolution operator of i-th qutrit initially in the state $|0\rangle_i$, which we write down in the form

$$\begin{aligned}\widehat{U}_i(t)\,|0\rangle_i \;\simeq\;& \mathrm{e}^{-\mathrm{i}u_i^+\int^t \mathrm{e}^{\mathrm{i}\alpha x}E_i(x)\mathrm{d}x-\mathrm{i}t_i^+\int^t \mathrm{e}^{\mathrm{i}\alpha x}E_i^*(x)\mathrm{d}x}\\ & \mathrm{e}^{-\int^t E_i(y)\int^y \mathrm{e}^{\mathrm{i}\alpha(y-x)}E_i^*(x)\mathrm{d}x\mathrm{d}y}\\ & \mathrm{e}^{-\int^t E_i^*(y)\int^y \mathrm{e}^{\mathrm{i}\alpha(y-x)}E_i(x)\mathrm{d}x\mathrm{d}y}\,|0\rangle_i\;.\end{aligned} \tag{9}$$

It is due to this very approximation that the problem becomes analytically soluble.

Substitution of (9) into (5) with the allowance for the relation $\sum_i V_k^{1/2}V_m^{1/2}C_{im}C_{ik}=\delta_{km}V_m$ yields the action

$$\begin{aligned}S \;=\;& \sum_m\int Z_m(t)Z_m^*(t)\mathrm{d}t\\ & -\sum_{j,m}V_m^{1/2}C_{jm}\int^t \mathrm{e}^{\mathrm{i}\alpha x}\left(u_j^+Z_m(x)+t_j^+Z_m^*(x)\right)\mathrm{d}x\\ & -\sum_m V_m\int^t\int^y \mathrm{e}^{\mathrm{i}\alpha(y-x)}Z_m(y)Z_m^*(x)\mathrm{d}x\mathrm{d}y\\ & -\sum_m V_m\int^t\int^y \mathrm{e}^{\mathrm{i}\alpha(y-x)}Z_m^*(y)Z_m(x)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{10}$$

which is bilinear in the field variables Z_m and Z_m^*. This allows one to exactly evaluate the Gaussian functional integral by performing standard calculations: from the action S of (10) one derives the Lagrange equations $\frac{\delta S}{\delta Z_m(x)}=0$, $\frac{\delta S}{\delta Z_m^*(x)}=0$ for the extremum trajectory. By substituting the solutions of these equations

$$\begin{aligned}Z_m(x) &= -\frac{(\mathrm{i}\alpha\sin x\omega_m+\omega_m\cos x\omega_m)\sum_i V_m^{1/2}C_{mi}t_i^+}{\mathrm{i}\,(\alpha+V_m)\sin t\omega_m+\omega_m\cos t\omega_m}\\ Z_m^*(x) &= \frac{(\mathrm{i}\alpha\sin x\omega_m+\omega_m\cos x\omega_m)\sum_i V_m^{1/2}C_{mi}u_i^+}{\mathrm{i}\,(\alpha+V_m)\sin t\omega_m+\omega_m\cos t\omega_m}\end{aligned}$$

with $\omega_m=\sqrt{\alpha\,(\alpha+2V_m)}$, to (10) we find the part of S which depends only on the operators u_i^+ and t_i^+ but not on the fields Z_m and Z_m^*. The remaining part depending on the fields but not on u_i^+ and t_i^+ is a functional integral which gives a c-number and thus can be ignored. The evaluation yields $\exp\left\{-\mathrm{i}\widehat{H}t\right\}|\mathsf{O}\rangle\sim\exp\left\{\sum_{i,,j=1}^{N}u_j^+W_{ij}t_i^+\right\}|\mathsf{O}\rangle$ with the tanglemeter matrix

$$\widehat{W}(t)=\frac{\widehat{V}}{\mathrm{i}\sqrt{\alpha(2\widehat{V}+\alpha)}\cot\left[t\sqrt{\alpha(2\widehat{V}+\alpha)}\right]-\widehat{V}-\alpha}\;. \tag{11}$$

Substitution of (11) to (2) yields

$$n_1=\mathrm{Tr}\frac{\widehat{V}^2\sin^2\left[t\sqrt{\alpha(2\widehat{V}+\alpha)}\right]}{\alpha(2\widehat{V}+\alpha)} \tag{12}$$

We are now in the position to consider several particular examples. We start with the case where all qutrits interact via a collective dipole moment, that is when $V_{i,j}=V=\mathrm{const}$. In this case the matrix $\widehat{W}$ has only one nonzero eigen value $V_1=NV$. This results in the upper state population

$$n_1=\frac{N^2V^2}{\alpha(2NV+\alpha)}\sin^2\left[t\sqrt{\alpha(2NV+\alpha)}\right]\;, \tag{13}$$

shown in figure 1 as a function of the collective coupling NV and the detuning α. The maximum rate of the creation of the qutrits in the upper states corresponds to $\alpha=-NV$ where $n_1=\sinh^2[NVt]$. This expression is valid as long as $\sinh^2[NVt]\ll N$, that is when the assumption *(i)* is fulfilled. One gets a deeper insight into the physical meaning of this result by comparing the assembly of collectively interacting qutrits with the lasing of an inverted two-level media [13]. In this case the operator t^+ corresponds to the photon creation operator $a\dagger$ while u^+ corresponds to the excitation annihilation operator σ^-. As long as the total number of the emitted photons remains much smaller than the total number of the two-level atoms these two systems are almost equivalent, – the main difference being that the collective dipole-dipole interaction shifts the resonance of the lasing rate from $\alpha=0$ to the point $\alpha=-NV$.

Next example is an individual dipole-dipole coupling of qutrits with $V_{ji}=\mu^2(1-2\cos^2\theta_{ji})/r_{ji}^3$, where r_{ji} is the distance between i-th and j-th qutrits, θ_{ji} is the angle between the radius-vector $\overrightarrow{r}_{ji}$ and the z direction, and μ is

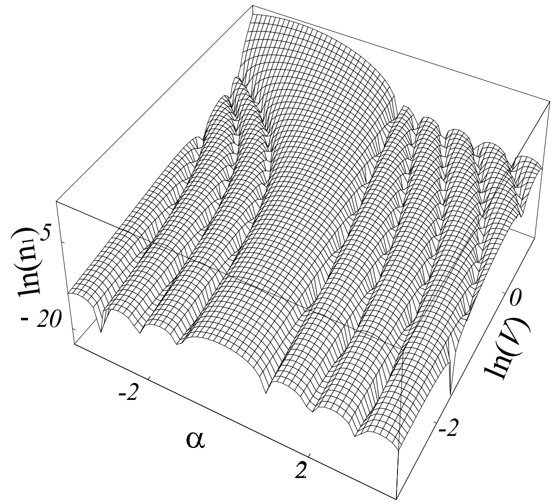

FIG. 1: Number n_1 of the qutrits in the upper state as a function of the coupling $v = NV$ and the detuning α for $t = 3$.

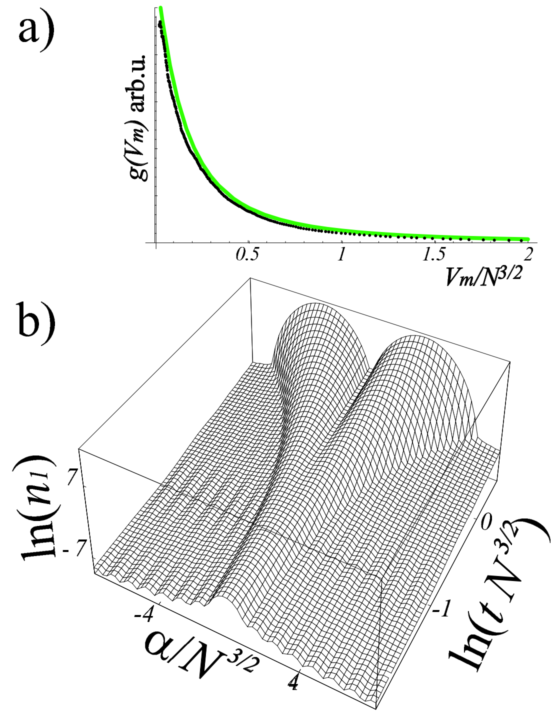

FIG. 2: a) Density of the eigenvalues V_m for $N = 300$ dipoles averaged over 100 random distributions in a unit cube (dots) and a heuristic fit $g(V_m) = N/4V_m e^{\sqrt{\pi/2}} \cosh \frac{\ln 4V_m}{\sqrt{\pi}}$ (solid green line). We set the dipole moment $\mu = 1$. The distribution is symmetric with respect to zero. b) Number n_1 of the qutrits in the upper state as a function of the scaled time $tN^{3/2}$ and the scaled detuning $\alpha/N^{3/2}$.

the dipole moment matrix element. We consider the position of each qutrit as an independent random variable and assume that the qutrits have uniform spatial density n. The statistical properties of the random matrix corresponding to $1/r^3$ interaction in disordered media is a challenging problem which has already been addressed in the context of spin glasses [14, 15] and cold Rydberg atoms [2, 5]. In particular, the distribution $g(V_m)$ of the eigenvalues of V_{ij} found numerically [16] has essentially non-analytical behavior near $V_m \to 0$, in contrast with the well-known Wigner semicircular distribution of the Gaussian random matrix eigenvalues. In figure 2 along with the results of a similar numerical work performed for a larger statistical ensemble and followed by a more accurate analytical fit to the distribution of V_m we depict the population (12) averaged over this distribution $g(V_m)$.

Note that the numerical results with $\mu = 1$ are given in heuristic dimensionless units $\alpha/N^{3/2}$. Yet unknown is the energy parameter correctly describing the cooperative phenomena in ensembles with $1/r^3$ interaction. We guess that it might resemble the combination of parameters $\mu^2 n\sqrt{N}$ where $\mu^2 n$ is the typical two-particle interaction while the typical parameter $\sqrt{N}$ allows for the cooperative effects resembling Dicke superadience [17] of two-level particles. However, unambiguous identification of this parameter is the subject of a more detailed future consideration. We also note that by assuming small deviations of the qutrits from their initial state, we discard from the consideration the strongly interacting dimers of qutrits that give an important contribution to n_1 [2] at large α. The latter effect requires a more sophisticated model, which would consider these dimers as the assembly elements of another type.

The third example concerns a controlled behavior of the assembly where, as earlier, the qutrits interact via the collective dipole, but variation of α is now possible during this interaction. This example shows that the behavior of the three-level systems, though similar, still is richer than that of the two-level systems and may display effects similar to mode beats. We consider the case where α, initially different from zero, can be switched off at $t = t_1$, then remains zero during a time interval $\Delta t = t_2 - t_1$, and takes the initial value for $t > t_2$. We notice the change of the symmetry associated with such a control: a three-level quantum system, which generically has the $su(3)$ symmetry, at $\alpha = 0$ becomes equivalent to a spin-1 particle possessing the $su(2)$ symmetry. For the description, in the equation (10) for the action one has to replace $\mathrm{e}^{\mathrm{i}\alpha x}$ and $\mathrm{e}^{\mathrm{i}\alpha(y-x)}$ by $\mathrm{e}^{\mathrm{i}\int_0^x \alpha(y)\mathrm{d}y}$ and $\mathrm{e}^{\mathrm{i}\int_x^y \alpha(s)\mathrm{d}s}$, respectively.

When $V \ll \alpha$, the solution of the integral Langrange equations results in

$$n_1 \simeq \frac{V^2}{\alpha^2} \sin^2 \left[(V + \alpha)(t - \Delta t) \right] \tag{14}$$

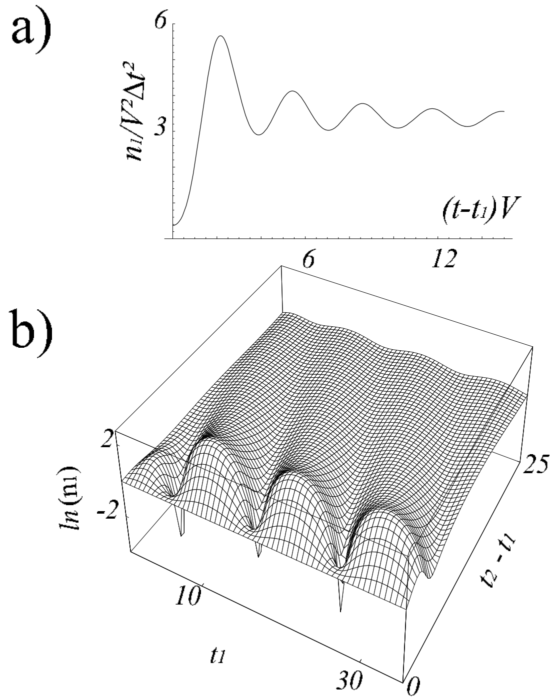

FIG. 3: (a) Number of the particles in the upper state n_1 for time dependent α. We set $\alpha = 0$ when $t_1 < t < t_2$ and the typical coupling $V \simeq \mu^2 n\sqrt{N}$. (b) Ratio $n_1(NV, \alpha, t, t_1, t_2)/n_1(NV, \alpha, t, t_1, t_1)$ of the numbers of the qutrits in the upper state as a function of the scaled time t_1 and the time difference $t_2 - t_1$. We set $t = t_2 + 0.9$; $NV = -\alpha = 0.3$.

for $\alpha\Delta t \ll 1$ and in

$$n_1 \simeq V^2\Delta t^2 \{1 + 2[1 - \cos V(t - t_2)][1 + \cos\alpha(t_1 + t_2)]\} \tag{15}$$

for $\alpha\Delta t \gg 1$. One can eliminate the second brackets in (15) by averaging out the rapid oscillations at the frequency α. In figure 3(a) we depict $n_1(t - t_2)$ averaged not only over these oscillations but also over the distribution of the collective couplings $g(V_m)$ presented in figure 2. One can see that the population displays dying oscillations as a function of the variable $t - t_2$ that is the time elapsed after the moment t_2 when the detuning α is switched back. These oscillations can be interpreted as beats of the symmetric $\frac{|-1\rangle+|1\rangle}{\sqrt{2}}$ and antisymmetric $\frac{|-1\rangle-|1\rangle}{\sqrt{2}}$ combinations of the upper and the lower states of qutrits in the interaction representation. As a direct consequence of $su(2)$ symmetry, the combination $\frac{|-1\rangle+|1\rangle}{\sqrt{2}}$ evolves while $\frac{|-1\rangle-|1\rangle}{\sqrt{2}}$ conserves when $\alpha = 0$. The analytical solution is straightforward– though cumbersome, and it does not contain any principal technical complications. In figure 3(b) we depict the ratio $n_1(NV, \alpha, t, t_1, t_2)/n_1(NV, \alpha, t, t_1, t_1)$ as a function of t_1 and t_2 obtained explicitly for $V = \text{const}$ with the help of Mathematica package. Again one can see the population oscillations resulting from the quantum beats.

We conclude by summarizing the results obtained. *(i)* The collective behavior of an assembly of three-level elements (qutrits) admits an exhaustive analytical description when each qutrit does not change significantly its initial quantum state. This description is based on the elements' dynamic separation with the help of a functional integral and invokes the time-dependent second-order perturbation theory followed by the exact evaluation of a Gaussian functional integral. *(ii)* In the simplest case where the interaction among the qutrits occurs via collective dipole-dipole interaction, the dynamics of the systems resembles the lasing of inverted two-level media: for the qutrits initially in the middle state the collective downward transitions are accompanied by the collective upward transitions similar to the photon creation in a coherent state. The only difference is that the collective dipole-dipole interaction shifts the resonance center such that the maximum transition rate occurs when the frequencies of the downward and upward transitions differ by the amount of the dipole collective coupling. *(iii)* This effect persists for the regular dipole-dipole interaction and yields a two-hump frequency dependence of the transition rate. Positions of the maxima are found with the help of the collective coupling distribution, which has been numerically obtained for the media with $1/r^3$ interaction. *(iv)* Still three-level systems may have a more complex behavior than two-level ones displaying quantum beats. This effect occurs when in the course of time one changes the position of the middle level relative to the positions of the upper and the lower levels switching in this way between the $su(2)$ and $su(3)$ symmetry of the qutrits.

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